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On the Existence of Functions with Prescribed Best Approximations

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1. INTRODUCTION AND PRELIMINARIES

In a conference held in Oberwolfach in 1968, Rivlin [7] posed the following problem:

> Characterize those n-tuples $\{p_0, p_1, ..., p_{n-1}\}$ of algebraic polynomials such that the degree of p_j is j, $j = 0, 1, ..., n - 1$, for which there exists a function $f \in C([a, b])$, the space of all continuous real valued functions on $[a, b]$, such that the polynomial of best approximation to f, in the sense of Chebyshev, of degree j, is p_j, for $j = 0, 1, ..., n - 1$.

Earlier, in 1957, Paszkowski [6] characterized two polynomials of successive degrees, with the above property. Deutsch, Morris, and Singer [3] have considered the above problem in a general normed linear space and have characterized a sequence of elements of linear subspaces for which there exists an element having the sequence of elements as best approximations in the corresponding subspaces. In particular, they have given a solution to Rivlin's problem for constant and linear functions. Sprecher [8] has considered two polynomials of arbitrary degrees and in [9] he has given a solution to the above problem for the case $n = 3$. Subrahmanya [10] has generalized the case $n = 2$ to a general Chebyshev system and in [11] has given a solution to the above problem for a general n. Hegering [5] has considered the above problem in normed linear spaces that include $C(T)$. In all the above papers, except that of Deutsch *et al.*, only a finite number of elements are considered.

In this paper we consider the above problem in $C(T)$, T compact and characterize an infinite set of elements for which there exists an element $f \in C(T)$ with this set as best approximations from arbitrary subsets which

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we assume suns only in the necessity part. This is given in Theorem 2. Our main result is given in Theorem 1 of this paper, from which we get a number of other corollaries, including generalizations of a particular case of a theorem of Brosowski [2], and a theorem of Subrahmanya [11].

Let T be a compact Hausdorff space and let $C(T)$ denote the set of all continuous real valued functions defined on T, normed by

$$\|f\| = \sup_{t\in T} |f(t)| .$$

Let $\varnothing \neq V \subset C(T)$ and $f \in C(T)$. An element $v_0 \in V$ is said to be a best approximation to f in V, if

$$\|f - v_0\| = \inf_{v\in V} \|f - v\| = E_V(f).$$

We denote by $P_V(f)$, the set of all best approximation to f in V, i.e.,

$$P_V(f) = \{v \in V \mid \|f - v\| = E_V(f)\}.$$

V is said to be a sun if whenever $v_0 \in P_V(f)$ for some $f \in C(T)$ implies

$$v_0 \in P_V(v_0 + \lambda(f - v_0))$$

for every $\lambda \geqslant 1$.

A signature ϵ on T is a continuous mapping of a closed subset of T into $\{-1,1\}$. The set of all signatures on T is denoted by SIG[T]. A signature ϵ is said to be extremal for the element v_0 (with respect to $V \subseteq C(T)$) if for every $v \in V$ we have

$$\min_{t\in \mathrm{DOM}(\epsilon)} \epsilon(t)(v(t) - v_0(t)) \leqslant 0$$

If $f \in C(T)$, we denote by M_f the following set:

$$M_f = \{t \in T \mid |f(t)| = \|f\|\}.$$

For $f \neq 0$, there is a natural signature ϵ_f defined by

$$\epsilon_f(t) \cdot f(t) = |f(t)| = \|f\|.$$

Then we have the following well-known result [1]:

LEMMA 1. *Let $V \subset C(T)$. Then V is a sun if and only if whenever $v_0 \in P_V(f)$, $f \in C(T)\setminus V$, implies ϵ_{f-v_0} is extremal for v_0.*

LEMMA 2. (a) *The mapping Φ: $C(T) \times T \to \mathbb{R}$ defined by $\Phi(f, t) = f(t)$ is continuous.*

(b) *For compact $A \subset C(T)$ the functions*

$$y(t) := \sup_{x \in A} x(t) \text{ and } z(t) := \inf_{x \in A} x(t)$$

are continuous on T.

Proof. (a) Let $f_0 \in C(T)$, $t_0 \in T$ and $\epsilon > 0$ be given. Then for every (f, t) in the open set

$$\{f \in C(T) \mid \|f - f_0\| < \epsilon/2\} \times \{t \in T \mid |f_0(t) - f_0(t_0)| < \epsilon/2\}$$

we have

$$|f(t) - f_0(t_0)| \leqslant |f(t) - f_0(t)| + |f_0(t) - f_0(t_0)| < \epsilon$$

which shows that Φ is continuous.

(b) Let (t_ν) be a net converging to $\bar{t}$. For each ν there exists an x_ν in A such that $x_\nu(t_\nu) = y(t_\nu)$ and an $\tilde{x}$ in A such $x(t) = y(\bar{t})$. Since A is compact we can assume that x_ν converges to x in A. By part (a) and by the continuity of $\bar{x}$ we can conclude from

$$\bar{x}(t_\nu) < y(t_\nu) = \Phi(x_\nu, t_\nu)$$

the inequality

$$\tilde{x}(\bar{t}) = \bar{x}(\bar{t}) \leqslant y(\bar{t}) = \tilde{x}(\bar{t})$$

and hence the continuity of y. By the same method one can prove the continuity of z.

Let K be a compact Hausdorff space and let $v: K \to C(T)$ and $e: K \to \mathbb{R}^+$ be continuous mappings. Then we define the continuous mapping

$$v^\delta: K \to C(T), \qquad \delta \in \{-1, 1\}$$

by $v^\delta(\kappa) := v_\kappa{}^\delta := v_\kappa + \delta e_\kappa := v(\kappa) + \delta e(\kappa)$.

We then set

$$v_{-1} = \sup_{\kappa \in K} v_\kappa^{-1} \qquad \text{and} \qquad v_{+1} = \inf_{\kappa \in K} v_\kappa^{+1}.$$

2. The Main Result

Theorem 1. *Let K be a compact Hausdorff space and let $v: K \to C(T)$ and $e: K \to \mathbb{R}^+$ be continuous mappings and let $V: K \to \mathrm{POT}\,(C(T))$ be such that, for all κ in K, we have $v_\kappa \in V_\kappa$. Then in order that there exists a function $f \in C(T)$ such that $v_\kappa \in P_{V_\kappa}(f)$ and $e_\kappa = \|f - v_\kappa\|$ it is sufficient and if for every $\kappa \in K$, V_κ is a sun then it is also necessary that there exists a mapping*

$\epsilon: K \to \mathrm{SIG}(T)$ such that ϵ_κ is extremal for v_κ(w.r.t. V_κ) *with the following properties*:

(i) *For every $t \in T$ we have $v_{-1}(t) \leqslant v_{+1}(t)$.*

(ii) *For every $\kappa \in K$ we have*

$$t \in \mathrm{DOM}(\epsilon_\kappa) \Rightarrow v_{\epsilon_\kappa(t)}(t) - v_\kappa^{\epsilon_\kappa(t)}(t) = 0.$$

(iii) *For every pair κ, $\mu \in K$ we have*

$$t \in \mathrm{DOM}(\epsilon_\kappa) \cap \mathrm{DOM}(\epsilon_\mu) \Rightarrow v_{\epsilon_\kappa(t)}(t) - v_{\epsilon_\mu(t)}(t) = 0.$$

(iv) *The set $K^\delta := \{\kappa \in K \mid \epsilon_\kappa^{-1}(\delta) \neq \varnothing\}$ is closed and the mapping $M^\delta: K^\delta \to 2^T$ defined by $\kappa \mapsto \epsilon^{-1}(\delta)$ is upper semicontinuous where*

$$\delta \in \{-1, +1\}.$$

Proof. Necessity of the conditions. For $f \in C(T)$ with the properties of the theorem define a mapping $\epsilon: K \to \mathrm{SIG}(T)$ by ϵ_{f-v_κ} for each $\kappa \in K$. Since each V_κ is a sun the signature ϵ_κ is extremal for v_κ. Since by assumption $v_\kappa \in P_{V_\kappa}(f)$ and $e_\kappa = \|f - v_\kappa\|$ we conclude from

$$-e_\kappa \leqslant f(t) - v_\kappa(t) \leqslant e_\kappa$$

for each $t \in T$ and each $\kappa \in K$ that

$$v_{-1}(t) \leqslant f(t) \leqslant v_1(t)$$

which implies condition (i). Every $t \in \mathrm{DOM}\,(\epsilon_\kappa)$ satisfies

$$f(t) - v_\kappa(t) = \epsilon_\kappa \|f - v_\kappa\| = \epsilon_\kappa e_\kappa\,.$$

Using the last inequality we conclude

$$v_\kappa^{\epsilon_\kappa(t)} = v_\kappa(t) + \epsilon_\kappa(t)\, e_\kappa = f(t) = v_{\epsilon_\kappa(t)}(t)$$

which proves (ii).

Every $t \in \mathrm{DOM}\,(\epsilon_\kappa) \cap \mathrm{DOM}(\epsilon_\mu)$ satisfies the equations

$$f(t) = v_\kappa(t) + \epsilon_\kappa(t)\, e_\kappa = v_{\epsilon_\kappa(t)}(t),$$

$$f(t) = v_\mu(t) + \epsilon_\mu(t)\, e_\mu = v_{\epsilon_\mu(t)}(t)$$

which imply condition (iii).

For $\delta \in \{-1, 1\}$ define the set

$$A^\delta := \{v_\kappa \in C(T) \mid \epsilon_\kappa^{-1}(\delta) \neq \varnothing \quad \text{and} \quad \kappa \in K\}$$

which is contained in the compact set IM(v), the image of v. Choose a net (v_{κ_n}) in A^δ converging to v_{k_0} in IM(v).

If $v_{\kappa_0} = f$, then $v_\kappa \in A^1 \cap A^{-1}$. If $v_{\kappa_0} \neq f$ then choose a subnet (v_{κ_λ}) of (v_{κ_n}) and a net (t_λ) in T converging to t_0 in T and $f(t_\lambda) - v_{\kappa_\lambda}(t_\lambda) = \delta \| x - v_{\kappa_\lambda} \|$. By Lemma 2, we have

$$\delta \| f - v_{\kappa_\lambda} \| = f(t_\lambda) - v_{\kappa_\lambda}(t_\lambda) \to f(t_0) - v_{\kappa_0}(t_0)$$

By the continuity of the norm we conclude that

$$\delta \| f - v_{\kappa_0} \| = f(t_0) - v_{\kappa_0}(t_0).$$

Hence, $v_{\kappa_0} \in A^\delta$, which shows that A^δ is closed. By the continuity of v the set K^δ is closed. Since the mapping $\kappa \to f - v_\kappa$, $\kappa \in K$, is continuous it suffices to prove that the mapping $g \to \epsilon_g^{-1}(\delta)$, $g \in C(T)$, is upper semicontinuous. If not there exist nets (g_λ) in $C(T)$ and t_λ in T converging to $g_0 \in C(T)$ and resp. to t_0 in T and an open set U_0 containing $\epsilon_{g_0}^{-1}(\delta)$ such that $t_\lambda \in \epsilon_{g_\lambda}^{-1}(\delta)$ and $U_0 \cap \{t_\lambda\} = \varnothing$. The last condition implies $t_0 \notin \epsilon_{g_0}^{-1}(\delta)$. By Lemma 2 we have $g_\lambda(t_\lambda) \to g_0(t_0)$. Since $\delta \| g_\lambda \| = g_\lambda(t_\lambda)$ we have by the continuity of the norm $\delta \| g_0 \| = g_0(t_0)$ and hence $t_0 \in \epsilon_{g_0}^{-1}(\delta)$, which is a contradiction. This proves condition (iv) and completes the proof of the necessity.

Sufficiency of the conditions. Since the mappings v and e are continuous the mapping $v^\delta\colon K \to C(T)$ is also continuous, $\delta \in \{-1, 1\}$. By Lemma 2, v_{-1} and v_1 are continuous functions on T.

By condition (iv) the set K^δ, $\delta \in \{-1, 1\}$, is closed and by compactness of K also compact. Since the mapping M^δ is upper semicontinuous by a theorem of Hahn [4] the set

$$N^\delta := \bigcup_{\kappa \in K} M^\delta(\kappa), \qquad \delta \in \{-1, +1\}$$

is compact. Now define a function

$$g\colon N^1 \cup N^{-1} \to \mathbb{R}$$

by $g(t) := v_\delta(t)$ for $t \in N^\delta$. Then g is well-defined by condition (iii) and it is continuous by the continuity of v_1, v_{-1}. By Tietze's theorem there exists a function $f \in C(T)$ such that $f(t) = g(t)$ for $t \in N^{1|} \cup N^{-1}$.

We can assume that $v_{-1}(t) \leqslant f(t) \leqslant v_1(t)$ for all $t \in T$. For, if not consider the function

$$\bar{f}(t) := \begin{cases} v_\delta(t) & \text{for} \quad t \in \{t \in T \mid \delta(v_\delta(t) - f(t)) \leqslant 0\} \\ f(t) & \text{for} \quad t \in \{t \in T \mid v_{-1}(t) \leqslant f(t) \leqslant v_1(t)\} \end{cases}$$

which is well-defined by condition (iii) and continuous (cf. [2, p. 38]). Further it follows that $\bar{f}(t) = g(t)$ for $t \in N^1 \cup N^{-1}$, $v_{-1}(t) \leqslant \bar{f}(t) \leqslant v_1(t)$ for all $t \in T$. Consequently, we have

$$-e_\kappa \leqslant f(t) - v_\kappa(t) \leqslant e_\kappa$$

for all $t \in T$, which implies $\|f - v\| \leqslant e_\kappa$. Also if $t \in \mathrm{DOM}(\epsilon_\kappa)$ then we have, from condition (ii),

$$v_{\epsilon_\kappa(t)}(t) - v_\kappa^{\epsilon_\kappa(t)}(t) = 0 \qquad \text{and} \qquad t \in N^{\epsilon_\kappa(t)}.$$

Consequently, $f(t) = v_{\epsilon_\kappa(t)}(t) = v_\kappa^{\epsilon_\kappa(t)}(t)$. That is

$$f(t) - v_\kappa(t) = \epsilon_\kappa(t)\, e_\kappa$$

which shows that $\mathrm{DOM}(\epsilon_\kappa) \subset \mathrm{DOM}(\epsilon_{f-v_\kappa})$ and $\epsilon_\kappa(t) = \epsilon_{f-v_\kappa}(t)$ for each t in DOM (ϵ_κ). Since ϵ_κ is extremal we can conclude that $v_\kappa \in P_{V_\kappa}(f)$. This completes the proof of the sufficiency of the conditions.

3. Some Corollaries

THEOREM 2. *Let K be a compact Hausdorff-space and let $v: K \to C(T)$ be a continuous mapping. Let $V: K \to \mathrm{POT}(C(T))$ be such that $v_\kappa \in V_\kappa$ for each $\kappa \in K$. Then in order that there exists an $f \in C(T)$ with $v_\kappa \in P_{V_\kappa}(f)$ it is sufficient and if for each $\kappa \in K$, V_κ is a sun then it is also necessary that there exist a continuous mapping $e: K \to \mathbb{R}^+$ and a mapping $\epsilon: K \to \mathrm{SIG}(T)$ such that ϵ_κ is extremal for v_κ with the following properties*

(i) $e_\mu + e_\kappa \geqslant \| v_\mu - v_\kappa \|$ *for each pair* $\kappa, \mu \in K$

(ii) $\min_{t \in \mathrm{DOM}(\epsilon_\kappa)} \epsilon_\kappa(t)(v_\mu(t) - v_\kappa(t)) \geqslant e_\kappa - e_\mu$ *for each pair* $\kappa, \mu \in K$

(iii) *as in Theorem* 1

(iv) *as in Theorem* 1.

Proof. Necessity of the conditions. Define a continuous mapping $e: K \to \mathbb{R}^+$ by $e_\kappa = \|f - v_\kappa\|$, $\kappa \in K$. By Theorem 1 there exists a mapping $\epsilon: K \to \mathrm{SIG}(T)$ with properties (i)–(iv) of Theorem 1. Condition (i) of the Theorem 2 is an immediate consequence of the triangle inequality. By (ii) of Theorem 1 we have for each $t \in \mathrm{DOM}(\epsilon_\kappa)$

$$v_{\epsilon_\kappa(t)}(t) - v_\kappa^{\epsilon_\kappa(t)}(t) = 0,$$

from which we conclude

$$\epsilon_\kappa(t)(v_\kappa(t) + \epsilon_\kappa(t)\, e_\kappa) \leqslant \epsilon_\kappa(t)(v_\mu(t) + \epsilon_\kappa(t)\, e_\mu)$$

for each $\mu \in K$, or

$$\min_{t \in \mathrm{DOM}(\epsilon_\kappa)} \epsilon_\kappa(t)(v_\mu(t) - v_\kappa(t)) \geqslant e_\kappa - e_\mu ,$$

which proves (ii).

Sufficiency of the conditions. Condition (i) of Theorem 2 implies

$$v_\mu(t) - v_\kappa(t) \leqslant e_\kappa + e_\mu$$
$$\text{or} \qquad v_\mu(t) - e_\mu \leqslant v_\kappa(t) + e_\mu$$

for every $t \in T$ and every pair κ, $\mu \in K$, from which we conclude $v_{-1}(t) \leqslant v_1(t)$ for each $t \in T$.

Condition (ii) implies

$$\epsilon_\kappa(t)(v_\mu(t) + \epsilon_\kappa(t)\, e_\mu) \geqslant \epsilon_\kappa(t)(v_\kappa(t) + \epsilon_\kappa(t)\, e_\kappa)$$
$$= \epsilon_\kappa(t)\, v_\kappa^{\epsilon_\kappa(t)}(t)$$

for each $t \in \mathrm{DOM}(\epsilon_\kappa)$ and each $\mu \in K$.

Consequently

$$\epsilon_\kappa(t)\, v_\kappa^{\epsilon_\kappa(t)}(t) = \epsilon_\kappa(t)\, v_{\epsilon_\kappa(t)}(t)$$

which implies condition (ii) of Theorem 1. Now by Theorem 1 we can conclude the existence of the function f.

COROLLARY 1. *Let $V_1 \subset V_2 \subset \cdots \subset V_n$ be a sequence of Haar-subspaces of $C[a, b]$ with $d_\nu := \dim V_\nu$, $\nu = 1, 2, \ldots, n$, and let $v_1, v_2, \ldots, v_n$ elements in $C[a, b]$ such that $v_\nu \in V_\nu$, $\nu = 1, 2, \ldots, n$.*

If there exists points

$$a \leqslant t_{\nu,0} < t_{\nu,1} < \cdots < t_{\nu,d_\nu} \leqslant b,$$

real numbers

$$e_1 > e_2 > \cdots > e_n \geqslant 0,$$
$$\text{and} \qquad \eta_1, \eta_2, \ldots, \eta_n \in \{-1, +1\}$$

such that for all ν, μ with $\nu \neq \mu$ we have

$$\min_{0 \leqslant \kappa \leqslant d_\nu} \eta_\nu(-1)^\kappa (v_\mu(t_{\nu,\kappa}) - v_\nu(t_{\nu,\kappa})) > e_\nu - e_\mu ,$$

then there exists an $f \in C[a, b]$ such that

$$v_\nu \in P_{V_\nu}(f),$$

$\nu = 1, 2, \ldots, n.$

Proof. There exist real number $\delta_{\nu,\kappa}$ such that the points

$$\bar{t}_{\nu,\kappa} := t_{\nu,\kappa} + \delta_{\nu,\kappa}$$

are different and are contained in $[a, b]$, and that we still have

$$\min_{0 \leqslant \kappa \leqslant d_\nu} \eta_\nu(-1)^\kappa (v_\mu(\bar{t}_{\nu,\kappa}) - v_\nu(\bar{t}_{\nu,\kappa})) > e_\nu - e_\mu .$$

Now define with $A := \max_{\mu,\kappa} \| v_\mu - v_\kappa \|$ real numbers $\bar{e}_\nu := e_\nu + A$, $\nu = 1, 2,..., n$. Then we have $\bar{e}_\mu + \bar{e}_\nu \geqslant \| v_\mu - v_\nu \|$ for all μ, ν. Since each V_ν is a Haar-subspace of dimension d_ν the mapping

$$\epsilon_\nu : \{t_{\nu,0}, t_{\nu,1},..., t_{\nu,d_\nu}\} \to \{-1, +1\}$$

defined by

$$\epsilon_\nu(t_{\nu,\kappa}) := \eta_\nu(-1)^\kappa$$

is an extremal signature for V_ν (compare Brosowski [2]).

Now $\epsilon_1, \epsilon_2,..., \epsilon_n$, $\bar{e}_1, \bar{e}_2,..., \bar{e}_n$, and $v_1, v_2,..., v_n$ satisfy conditions (i) and (ii) of Theorem 2. Condition (iii) is fullfilled since the points $\bar{t}_{\nu,\kappa}$ are different and condition (iv) is fulfilled since K is finite. Consequently, by Theorem 2 there exists an f in $C[a, b]$ with $v_\nu \in P_{V_\nu}(f)$, $\nu = 1, 2,..., n$.

COROLLARY 2. *Let $V_1 \subset V_2 \subset \cdots \subset V_n$ be a sequence of Haar-subspaces of $C[a, b]$ with $d_\nu := \dim V_\nu$, $\nu = 1, 2,..., n$ and let $v_1, v_2,..., v_n$ elements in $C[a, b]$ such that $v_\nu \in V_\nu$, $\nu = 1, 2,..., n$.*

If there exist an f in $C[a, b]$ such that $v_\nu \in P_{V_\nu}(f)$, $\nu = 1, 2,..., n$, then there exist points

$$a \leqslant t_{\nu,0} < t_{\nu,1} < \cdots < t_{\nu,d_\nu} \leqslant b,$$

$\nu = 1, 2,..., n$ and $\eta_1, \eta_2,..., \eta_n \in \{-1, +1\}$ such that for all $n \geqslant \mu > \nu \geqslant 1$ either

$$\eta_\nu(-1)^\kappa(v_\mu(t_{\nu,\kappa}) - v_\nu(t_{\nu,\kappa})) > 0$$

$\kappa = 0, 1,..., d_\nu$, or $v_\mu = v_\nu$.

Proof. We have

$$E_{V_1}(f) \geqslant E_{V_2}(f) \geqslant \cdots \geqslant E_{V_n}(f)$$

If $E_{V_\nu}(f) = E_{V_\mu}(f)$ then by the Haar-condition $v_\nu = v_\mu$. If $E_{V_\nu}(f) > E_{V_\mu}(f)$ then we consider the extremal signature ϵ_{f-v_ν}. For all $t \in \mathrm{DOM}(\epsilon_{f-v_\nu})$ we can conclude from condition (ii) of Theorem 1 that

$$\epsilon_{f-v_\nu}(t)\, v_\nu(t) + E_{V_\nu}(f) \leqslant \epsilon_{f-v_\nu}(t)\, v_\mu(t) + E_{V_\mu}(f)$$

and hence

$$\epsilon_{f-v_\nu}(t)(v_\mu(t) - v_\nu(t)) \geqslant E_{V_\nu}(f) - E_{V_\mu}(f) > 0.$$

By the alternation theorem there exists a number $\eta_\nu \in \{-1, +1\}$ and points

$$a \leqslant t_{\nu,0} < t_{\nu,1} < \cdots < t_{\nu,d_\nu} \leqslant b,$$

such that

$$\epsilon_{f-v_\nu}(t_{\nu,\kappa}) = \eta_\nu(-1)^\kappa,$$

$\kappa = 0, 1,..., d_\nu$, where the points $t_{\nu,\kappa}$ can be chosen independently of μ.

COROLLARY 3. *Let $V_1 \subseteq V_2$ be Haar-subspaces of $C[a, b]$, $v_1 \in V_1$, and $v_2 \in V_2$.*

In order that there exists an element f in $C[a, b]$ such that $v_i \in P_{V_i}(f)$, $i = 1, 2$, it is necessary and sufficient that $v_1 - v_2$ has $d_1 :=$ dim V_1 zeros in the open interval (a, b) or is identically zero.

Proof. If there exists an f with this properties then by Corollary 2 there exist points

$$a \leqslant t_{1,0} < t_{1,1} < \cdots < t_{1,d_1} \leqslant b$$

and $\eta_1 \in \{-1, +1\}$ such that either

$$\eta_1(-1)^\kappa(v_2(t_{\nu,\kappa}) - v_1(t_{\nu,\kappa})) > 0,$$

$\kappa = 0, 1, 2,..., d_1$, or $v_2 = v_1$. From this we conclude that $v_2 - v_1$ has either d_1 zeros in (a, b) or is identically zero.

If there exist v_1, v_2 satisfying the condition of the corollary, then we can omit the case $v_1 = v_2$ because it is trivial. If $v_1 \neq v_2$ then let

$$a < \tau_1 < \tau_2 < \cdots < \tau_{d_1} < b$$

be the zeros of $v_1 - v_2$, put $\tau_0 := a$, and $\tau_{d_1+1} := b$. Then choose an

$$\eta_1 \in \{-1, +1\}$$

and for $\nu = 0, 1,..., d_1$ points $t_{1,\nu}$ in the open interval $(\tau_\nu, \tau_{\nu+1})$ such that

$$\eta_1(-1)^\kappa(v_2(t_{1,\kappa}) - v_1(t_{1,\kappa})) \geqslant \alpha > 0,$$

$\kappa = 0, 1,..., d_1$. Since $v_2 - v_1$ has at least one zero in (a, b) we can choose points

$$a \leqslant t_{2,0} < t_{2,1} < \cdots < t_{2,d_2} \leqslant b$$

such that $|v_2(t_{2,\kappa}) - v_1(t_{2,\kappa})| < \alpha/2$ for $\kappa = 0, 1,..., d_2$. With the real numbers $e_1 := \alpha_1$, $e_2 := \alpha/2$, and $\eta_2 = 1$ we have

$$\eta_1(-1)^\kappa(v_2(t_{1,\kappa}) - v_1(t_{1,\kappa})) > e_1 - e_2,$$

$\kappa = 0, 1,..., d_1$, and

$$\eta_2(-1)^\kappa(v_1(t_{2,\kappa}) - v_2(t_{2,\kappa})) > e_2 - e_1,$$

$\kappa = 0, 1,..., d_2$.

By Corollary 1 there exists an f such that $v_i \in P_{V_i}(f)$, $i = 1, 2$.

COROLLARY 4. *Let K be a compact set and $v: K \to C(T)$ be a constant mapping, say $v_\kappa = v$ for all $\kappa \in K$, and let $V: K \to \mathrm{POT}(C(T))$ be such that $v \in V_\kappa$ for all $\kappa \in K$. Then in order that there exists an $f \in C(T)\backslash V_\kappa$ with $v \in P_{V_\kappa}(f)$ for all $\kappa \in K$, it is sufficient and if V_κ is a sun for each $\kappa \in K$, it is also necessary that there exists a signature ϵ which is extremal for v with respect to V_κ for each $\kappa \in K$.*

COROLLARY 5. *Let K, v and V be as in theorem 1. Then in order that there exists an $f \in C(T)$ with $v_\kappa \in P_{V_\kappa}(f)$ and $\|f - v_\kappa\|$ is constant, it is sufficient and if V_κ is a sun for every $\kappa \in K$ it is also necessary that the conditions of the theorem are satisfied with*

$$v_{-1} := \min_{\kappa \in K} v_\kappa \qquad \text{and} \qquad v_{+1} := \max_{\kappa \in K} v_\kappa .$$

COROLLARY 6. *Let $V_{\kappa_1}, V_{\kappa_2},..., V_{\kappa_n}$ be subsets of $C(T)$, $v_{\kappa_i} \in V_{\kappa_i}$, $i = 1, 2,..., n$ and $e_{x_i} \geqslant 0$, $i = 1, 2,..., n$, be given. In order that there exists an $f \in C(T)$ such that $v_{\kappa_i} \in P_{V_{\kappa_i}}(f)$ with $\|f - v_{\kappa_i}\| = e_{\kappa_i}$ it is sufficient and if each V_{κ_i}, $i = 1, 2,..., n$, is a sun, it is also necessary that the following conditions are satisfied*:

(i) $v_1(t) \geqslant v_{-1}(t)$ *for each $t \in T$.*

(ii) *For each i, there exists an extremal signature ϵ_{κ_i} for v_{κ_i} such that if $t \in \mathrm{DOM}(\epsilon_{\kappa_i})$, then*

$$v_{\epsilon_{\kappa_i}(t)}(t) - v_{\kappa_i}^{\epsilon_{\kappa_i}(t)}(t) = 0.$$

(iii) *If $t \in \mathrm{DOM}(\epsilon_{\kappa_i}) \cap \mathrm{DOM}(\epsilon_{\kappa_j})$, then*

$$v_{\epsilon_{\kappa_i}(t)}(t) - v_{\epsilon_{\kappa_j}(t)}(t) = 0.$$

Remark. Corollary 6 generalizes a result of Subrahmanya [11].

COROLLARY 7. *Let K be a compact set, $v: K \to C(T)$ be a continuous mapping and $V \subset C(T)$ be such that $v_\kappa \in V$ for all $\kappa \in K$. Then in order that there exists an $f \in C(T)$ such that $v_\kappa \in P_V(f)$ for all $\kappa \in K$, it is sufficient and if V is a sun it is also necessary that there exists an extremal signature ϵ for some v_{κ_0} (and hence for all $\kappa \in K$) satisfying the following condition*:

$$v_\kappa(t) - v_{\kappa_0}(t) = 0 \text{ for all } t \in \mathrm{DOM}(\epsilon) \text{ and all } \kappa \in K.$$

Proof. If there exists an $f \in C(T)$ with $v_\kappa \in P_{V_\kappa}(f)$ for al $\kappa \in K$, then setting $\| f - v_\kappa \| = e$, we have the conditions the Theorem 1 with

$$V: K \to \mathrm{POT}(C(T))$$

and $e: K \to \mathbb{R}^+$ now constant mappings. Now by "the intersection theorem" (see [2, Satz 3.7]) we have

$$\epsilon = \bigcap_{\kappa \in K} \epsilon_{f - v_\kappa}$$

is extremal for v_κ for all $\kappa \in K$. If $t \in \mathrm{DOM}(\epsilon)$ then from condition (ii) of Theorem 1 it follows that

$$v_{\epsilon(t)}(t) - v_{\kappa_0}^{\epsilon(t)}(t) = 0$$

and

$$v_{\epsilon(t)}(t) - v_\kappa^{\epsilon(t)}(t) = 0$$

and hence it follows that $v_\kappa(t) - v_{\kappa_0}(t) = 0$ for all $t \in \mathrm{DOM}(\epsilon)$ and for all $\kappa \in K$, which proves that the condition is necessary. On the other hand let ϵ be extremal for v_{κ_0} and $v_\kappa(t) - v_{\kappa_0}(t) = 0$ for all $t \in \mathrm{DOM}(\epsilon)$ and all $\kappa \in K$.

Choose an e such that

$$2e \geqslant \| \max_{\kappa \in K} v_\kappa - \min_{\kappa \in K} v_\kappa \|$$

Then conditions (i) and (ii) of Theorem 1 are immediately satisfied by noting that ϵ is extremal for v_κ for all $\kappa \in K$, follows from the given condition. Also conditions (iii) and (iv) do not contribute and hence the proof is completed.

Remark. The necessity part of Corollary 7 is a generalization of a result (see [2, Satz 3.7]) of Brosowski.

COROLLARY 8. *Let $K = \{1, 2\}$, v and V be as in Theorem 1 be such that V_1 and V_2 are suns and $V_1 \subset V_2$. Then if there exists an $f \in C(T)$ with $v_i \in P_{V_i}(f)$ then there exists an extremal signature ϵ for v_1 satisfying the following condition*:

For all $t \in \mathrm{DOM}(\epsilon)$ we have either $\epsilon(t)(v_2(t) - v_1(t)) > 0$ or

$$v_2(t) - v_1(t) = 0.$$

Proof. If $E_{V_1}(f) = E_{V_2}(f)$ then from Corollary 7, there exists an extremal signature ϵ for v_1 with respect to V_2 and hence with respect to V_1 such that for all $t \in \mathrm{DOM}(\epsilon)$ we have

$$v_1(t) - v_2(t) = 0.$$

On the other hand if $E_{V_1}(f) > E_{V_2}(f)$, then since V_1 is a sun ϵ_{f-v_1} is extremal for v_1 (w.r.t. V_1) and from condition (ii) of the theorem, if $t \in \mathrm{DOM}(\epsilon_{f-v_1})$ and $\epsilon_{f-v_1}(t) > 0$, we have

$$v_1(t) + E_{V_1}(f) \leqslant v_2(t) + E_{V_2}(f)$$

and hence $\epsilon_{f-v_1}(t)(v_2(t) - v_1(t)) \geqslant E_{V_1}(f) - E_{V_2}(f) > 0$, and if $\epsilon_{f-v_1}(t) < 0$, then, again from condition (ii) of Theorem 1, we have

$$v_1(t) - E_{V_1}(f) \geqslant v_2(t) - E_{V_2}(f)$$

and hence $\epsilon_{f-v_1}(t)(v_2(t) - v_1(t)) \geqslant E_{V_1}(f) - E_{V_2}(f) > 0$. Which shows that for all $t \in \mathrm{DOM}(\epsilon_{f-v_1})$, we have

$$\epsilon_{f-v_1}(t)(v_2(t) - v_1(t)) > 0$$

and hence completes the proof.

Remark. Corollary 8 generalizes the necessity part of a result of Paszkowski [6] for the case of two polynomials of consecutive degrees (see also [8, 3, 10]). In all the abovecited papers, the condition was also sufficient. It seems that the condition of Corollary 8 is not sufficient, but we are unable to construct a counterexample.

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